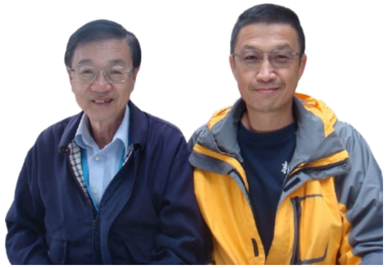
# Chapter 6: Maxwell Equations, Macroscopic Electromagnetism, Conservation Laws



## 6.1 Maxwell's Displacement Current; Maxwell Equations

**The Displacement Current :**

So far, we have the following set of laws :

$$\nabla\cdot\mathbf{D}=\rho_{free},\ \nabla\times\mathbf{H}=\mathbf{J}_{free},\ \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t},\ \text{and}\ \nabla\cdot\mathbf{B}=0 \quad (6.1)$$

Taking the divergence of $\nabla\times\mathbf{H}=\mathbf{J}_{free}$, we obtain

$$\underbrace{\nabla\cdot\nabla\times\mathbf{H}}_{0}=\nabla\cdot\mathbf{J}_{free}=0 \quad (6.2)$$

$$\Rightarrow\ \nabla\cdot\mathbf{J}_{free}+\frac{\partial\rho_{free}}{\partial t}\neq 0\ \text{if}\ \frac{\partial\rho_{free}}{\partial t}\neq 0$$

This violates the law of conservation of charge.

Maxwell observed that if we postulate

$$\nabla \times \mathbf{H} = \mathbf{J}_{free} + \frac{\partial \mathbf{D}}{\partial t}, \qquad (6.5)$$

where $\mathbf{J}_D \equiv \frac{\partial \mathbf{D}}{\partial t}$ is called the <u>displacement current</u> by Maxwell,

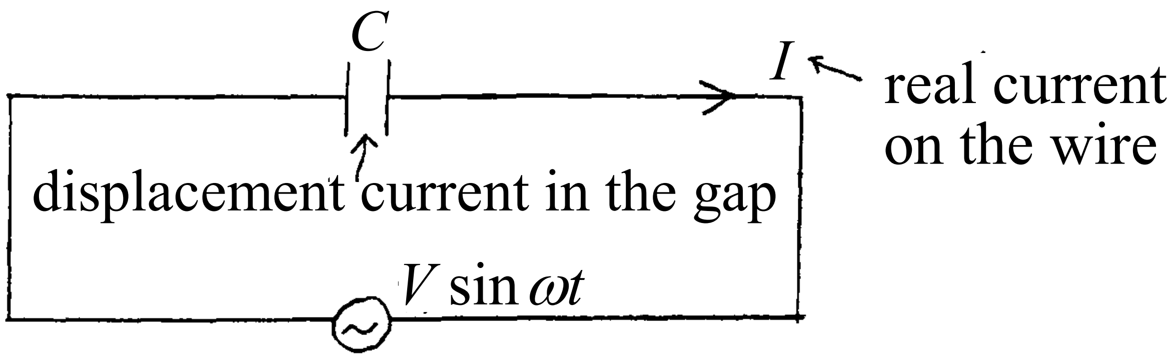
then, $\underbrace{\nabla \cdot \nabla \times \mathbf{H}}_{0} = \nabla \cdot \mathbf{J}_{free} + \frac{\partial}{\partial t} \nabla \cdot \mathbf{D} \Rightarrow \nabla \cdot \mathbf{J}_{free} + \frac{\partial \rho_{free}}{\partial t} = 0,$

which is consistent with the conservation of charge.

(6.5) can be written: $\nabla \times \mathbf{H} = \mathbf{J}_{free} + \mathbf{J}_D,$

The immediate significance of (6.5) is that it establishes a new mechanism to generate the **B**-field, i.e., by a time-varying **E**-field.

*Example of the displacement current*:



**The Maxwell Equations :**

In (6.1), replacing $\nabla \times \mathbf{H} = \mathbf{J}_{free}$ with $\nabla \times \mathbf{H} = \mathbf{J}_{free} + \frac{\partial \mathbf{D}}{\partial t}$, we have a new set of equations called the Maxwell equations:

$$\begin{cases} \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \\ \nabla \cdot \mathbf{D} = \rho_{free} \\ \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free} \end{cases} \tag{6.6}$$

homogeneous equations

inhomogeneous equations

These 4 equations form the basis of all classical electromagnetic phenomena. As discussed in Ch. 5, Faraday's law connects **E** and **B**. As will be shown in Ch. 7, (6.6) lead to EM waves. Thus, Maxwell's theory connects "optics" and "electromagnetism". On the other hand, the Lorentz force equation, $\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}$, connects "mechanics" and "electromagnetism".

## Review of Laws & Equations Obtained under Static Conditions :

*Scalar and vector potentials*:

$$\begin{cases} \nabla\times\mathbf{E}=0 & \text{(c)} \to \mathbf{E}=-\nabla\Phi, \\ \nabla\cdot\mathbf{E}=\dfrac{\rho}{\varepsilon_0} & \text{(b)} \to \nabla^2\Phi=-\dfrac{\rho}{\varepsilon_0}, \end{cases} \Rightarrow \quad \Phi=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x',$$

$$\begin{cases} \nabla\cdot\mathbf{B}=0 & \text{(e)} \to \mathbf{B}=\nabla\times\mathbf{A}, \\ \nabla\times\mathbf{B}=\mu_0\mathbf{J} & \text{(f)} \to \nabla^2\mathbf{A}=-\mu_0\mathbf{J} \end{cases} \Rightarrow \quad \mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x',$$

(with $\nabla\cdot\mathbf{A}=0$)

*Physical laws*:

$$\mathbf{E}=-\nabla\Phi=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}d^3\mathbf{x}' \quad \text{(a)} \quad \text{(pp. 27-30)}$$

$$\mathbf{B}=\nabla\times\mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}d^3\mathbf{x}' \quad \text{(d)} \quad \text{(pp. 178-9)}$$

*Question*: Which of the above laws/equations still hold true if $\dfrac{\partial}{\partial t}\neq 0$?

Why?

*Field energy*:

$$W_E = \frac{1}{2}\int \mathbf{E}\cdot\mathbf{D}\,d^3x \qquad (4.89)$$

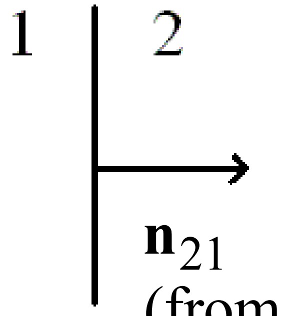
$$W_B = \frac{1}{2}\int \mathbf{B}\cdot\mathbf{H}\,d^3x \qquad (5.148)$$

*Forces*: $\mathbf{f} = \rho\mathbf{E} + \mathbf{J}\times\mathbf{B}$

$$\mathbf{F}_E = \int \rho\mathbf{E}\,d^3x$$

$$\mathbf{F}_B = \int \mathbf{J}\times\mathbf{B}\,d^3x$$

*Boundary conditions*:




(from region 1 to region 2)

$$\begin{cases} (\mathbf{D}_2 - \mathbf{D}_1)\cdot\mathbf{n}_{21} = \sigma_{free} \\ (\mathbf{E}_2 - \mathbf{E}_1)\times\mathbf{n}_{21} = 0 \end{cases} \qquad (4.40)$$

$$\begin{cases} (\mathbf{B}_2 - \mathbf{B}_1)\cdot\mathbf{n}_{21} = 0 & (5.86) \\ \mathbf{n}_{21}\times(\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}_{free} & (5.87) \end{cases}$$

*Question*: Which of the above equations still hold true if $\frac{\partial}{\partial t} \neq 0$? Why?

# 6.2 Vector and Scalar Potentials

From the 2 homogeneous Maxwell equations, we may find a vector potential **A** and a scalar potential $\Phi$ to represent **E** and **B**.

$$\nabla \cdot \mathbf{B} = 0 \qquad \Rightarrow \mathbf{B} = \nabla \times \mathbf{A} \qquad (6.7)$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad \Rightarrow \nabla \times \left( \mathbf{E} + \frac{\partial}{\partial t} \mathbf{A} \right) = 0 \; \Rightarrow \; \mathbf{E} + \frac{\partial}{\partial t} \mathbf{A} = -\nabla \Phi$$

$$\Rightarrow \mathbf{E} = -\nabla \Phi - \frac{\partial}{\partial t} \mathbf{A} \qquad (6.9)$$

With (6.7) and (6.9), we write the 2 inhomogeneous Maxwell equations (for *vacuum medium*) in terms of **A** and $\Phi$ as follows

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \Rightarrow \nabla^2 \Phi + \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) = -\frac{\rho}{\varepsilon_0} \qquad (6.10)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad \Rightarrow \nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{A} - \nabla \left( \nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial}{\partial t} \Phi \right)$$

$$\boxed{c^2 = \frac{1}{\varepsilon_0 \mu_0} \text{ in vacuum}} \qquad = -\mu_0 \mathbf{J} \qquad (6.11)$$

Thus, the set of 4 Maxwell equations for **E** and **B** have been reduced to 2 coupled equations for **A** and $\Phi$.

Rewrite
$$\begin{cases} \nabla^2\Phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\varepsilon_0} & (6.10) \\ \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} - \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi\right) = -\mu_0\mathbf{J} & (6.11) \end{cases}$$

If the potentials $\mathbf{A}$ and $\Phi$ satisfy the Lorenz condition:

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi = 0, \qquad (6.14)$$

then, (6.10) and (6.11) are uncoupled to give the equations:

$$\begin{cases} \nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi = -\frac{\rho}{\varepsilon_0} & (6.15) \\ \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} = -\mu_0\mathbf{J} & (6.16) \end{cases}$$

Equations (6.15) and (6.16), under the Lorenz condition, are equivalent in all respects to the Maxwell equations.

If $\mathbf{A}$ and $\Phi$ do not satisfy the Lorenz condition, then through the gauge transformation discussed below, we may obtain a new set of potentials $\mathbf{A}'$ and $\Phi'$, which satisfy the Lorenz condition.

# 6.3 Gauge Transformations, Lorenz Gauge, Coulomb Gauge

**Gauge Transformations :**

Rewrite (6.7) and (6.9):
$$\begin{cases} \mathbf{B} = \nabla \times \mathbf{A} & (6.7) \\ \mathbf{E} = -\nabla \Phi - \frac{\partial}{\partial t} \mathbf{A} & (6.9) \end{cases}$$

If $(\mathbf{A}, \Phi)$ are transformed to $(\mathbf{A}', \Phi')$ according to

$$\begin{cases} \mathbf{A}' = \mathbf{A} + \nabla \Lambda & (6.12) \\ \Phi' = \Phi - \frac{\partial}{\partial t} \Lambda & (6.13) \end{cases}$$

$\Lambda$ : an arbitrary scalar function of $\mathbf{x}$ and $t$

then $\mathbf{A}'$ and $\Phi'$ will give the same $\mathbf{E}$ and $\mathbf{B}$, i.e.,

$$\begin{cases} \mathbf{B} = \nabla \times \mathbf{A}' \\ \mathbf{E} = -\nabla \Phi' - \frac{\partial}{\partial t} \mathbf{A}' \end{cases}$$

The transformation defined by (6.12) and (6.13) is called the gauge transformation. The invariance of $\mathbf{E}$ and $\mathbf{B}$ under such transformations is called gauge invariance.

**Lorenz Gauge:**

Any set of $\mathbf{A}'$ and $\Phi'$ under the gauge tranformation gives the same **E** and **B**. Hence,

$$\begin{cases} \nabla^2\Phi' + \frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}') = -\frac{\rho}{\varepsilon_0} & (6.10) \\ \nabla^2\mathbf{A}' - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A}' - \nabla\left(\nabla\cdot\mathbf{A}' + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi'\right) = -\mu_0\mathbf{J} & (6.11) \end{cases}$$

If the original ($\mathbf{A}$, $\Phi$) do not satisfy the Lorenz condition, we may choose a gauge function $\Lambda$ and demand that the new ($\mathbf{A}'$, $\Phi'$) satisfy:

$$\nabla\cdot\mathbf{A}' + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi' = 0 \qquad (1)$$

This then uncouples $\mathbf{A}'$ and $\Phi'$ to give the same equations as in (6.15) and (6.16):

$$\begin{cases} \nabla^2\Phi' - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi' = -\frac{\rho}{\varepsilon_0} & (6.15) \\ \nabla^2\mathbf{A}' - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A}' = -\mu_0\mathbf{J} & (6.16) \end{cases}$$

Using $\mathbf{A}' = \mathbf{A} + \nabla\Lambda$ and $\Phi' = \Phi - \frac{\partial}{\partial t}\Lambda$, we obatin from (1) the equation for $\Lambda$: $\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Lambda = -\nabla\cdot\mathbf{A} - \frac{1}{c^2}\frac{\partial}{\partial t}\Phi \qquad (6.18)$

Rewrite (6.18): $$\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Lambda = -\nabla\cdot\mathbf{A} - \frac{1}{c^2}\frac{\partial}{\partial t}\Phi \qquad (6.18)$$

If ($\mathbf{A}$, $\Phi$) already satisfy the Lorenz condition, a restricted gauge transformation with $\Lambda$ given by the equation:

$$\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Lambda = 0 \qquad (6.20)$$

can preserve the Lorenz condition.

All ($\mathbf{A}$, $\Phi$) in this restricted class are said to belong to the Lorenz gauge. The Lorenz gauge is commonly used because it gives the set of equations [(6.15) and (6.16)] which treat $\mathbf{A}$ and $\Phi$ on equal footings. Furthermore, as will be shown in Eqs. (39) and (40) of Ch. 11, (6.15) and (6.16) as well as the Lorenz condition have the same form in all inertial frames.

Ludvig Valentin Lorenz: Danish physicist and mathematician, the Lorenz gauge.
Hendrik Antoon Lorentz: Dutch physicist (1902 Nobel Prize in Physics) Zeeman effect, the Lorentz transformation, and the Lorentz force.

Griffiths

## 10.1.3 Coulomb Gauge and Lorentz Gauge

There are many famous gauges in the literature. We will show the two most popular ones.

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{1}{\varepsilon_0}\rho$$

$$\left(\nabla^2 \mathbf{A} - \mu_0\varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \nabla\left(\nabla \cdot \mathbf{A} + \mu_0\varepsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \mathbf{J}$$

**The Coulomb Gauge**: $\nabla \cdot \mathbf{A} = 0$

$$\nabla^2 V = -\frac{1}{\varepsilon_0}\rho \text{ (Poisson's equation)}$$

$$V(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}',t)}{\mathscr{r}} d\tau' \quad \text{(setting } V = 0 \text{ at infinity)}$$

$V$ instantaneously reflects all changes in $\rho$. Really?

$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$ unlike electrostatic case.

Griffiths

# The Coulomb Gauge

***Advantage***: the scalar potential is particularly simple to calculate;

$$\nabla^2 V = -\frac{1}{\varepsilon_0}\rho \text{ (Poisson's equation)}$$

$$V(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}',t)}{\mathscr{r}} d\tau' \quad \text{(setting } V=0 \text{ at infinity)}$$

***Disadvantage***: the vector potential will be very difficult to calculate for the non-static case.

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} + (\mu_0\varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} + \nabla(\mu_0\varepsilon_0 \frac{\partial V}{\partial t}))$$

The Coulomb gauge is suitable for the static case.

Griffiths

# The Lorentz Gauge

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{1}{\varepsilon_0}\rho$$

$$\left(\nabla^2 \mathbf{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \nabla\left(\nabla \cdot \mathbf{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \mathbf{J}$$

**The Lorentz Gauge**: $\nabla \cdot \mathbf{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} = 0$

$$\nabla^2 V - \mu_0 \varepsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{1}{\varepsilon_0}\rho$$

$$\nabla^2 \mathbf{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}$$

inhomogeneous wave equations

$$\nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \equiv \Box^2$$

$\Box^2$: the d'Alembertian

$$\Box^2 V = -\frac{1}{\varepsilon_0}\rho$$

$$\Box^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

Griffiths

# The Lorentz Gauge

***Advantage***: It treats $V$ and $\mathbf{A}$ on an equal footing and is particularly nice in the context of special relativity. It can be regarded as four-dimensional versions of Poisson's equation.

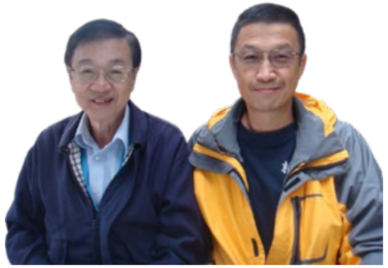
$V$ and $\mathbf{A}$ satisfy the *inhomogeneous wave equations*, with a "source" term on the right.

$$\Box^2 V = -\frac{1}{\varepsilon_0}\rho$$

$$\Box^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

***Disadvantage***: …

We will use the Lorentz gauge exclusively.



## Formal Solution of Electrostatic Boundary - Value Problem :

The expression $\Phi(\mathbf{x}) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int \dfrac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x'$ is applicable only to unbounded space. By Green's theorem, we may generalize it to an expression for bounded space with prescribed boundary conditions.

Consider a general electrostatic boundary-value problem:

$$\nabla^2\Phi(\mathbf{x}') = -\rho(\mathbf{x}')/\varepsilon_0 \quad \text{with } \Phi(\mathbf{x}') = \Phi_s(\mathbf{x}') \text{ for } \mathbf{x}' \text{ on } S \qquad (10)$$

Green's 2nd identity:


$$\int_V \left[\phi(\mathbf{x}')\nabla'^2\psi(\mathbf{x}') - \psi(\mathbf{x}')\nabla'^2\phi(\mathbf{x}')\right] d^3x'$$

$$= \oint_S \left[\phi(\mathbf{x}')\frac{\partial}{\partial n'}\psi(\mathbf{x}') - \psi(\mathbf{x}')\frac{\partial}{\partial n'}\phi(\mathbf{x}')\right] da' \qquad (1.35)$$

In (1.35), let $\phi(\mathbf{x}')$ be the solution of (10) with variable $\mathbf{x}'$ (i.e., $\Phi(\mathbf{x}')$). Let $\psi(\mathbf{x}') = G_D(\mathbf{x}, \mathbf{x}')$, where $G_D(\mathbf{x}, \mathbf{x}')$ is the Green function satisfying

$$\nabla'^2 G_D(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}') \quad \text{with } G_D(\mathbf{x},\mathbf{x}') = 0 \text{ for } \mathbf{x}' \text{ on } S \qquad (11)$$

Substitution of $\phi(\mathbf{x}')$ and $\psi(\mathbf{x}')$ into (1.35) gives

$$\int_v [\Phi(\mathbf{x}')\overbrace{\nabla'^2 G_D(\mathbf{x},\mathbf{x}')}^{-4\pi\delta(\mathbf{x}-\mathbf{x}')} - G_D(\mathbf{x},\mathbf{x}')\overbrace{\nabla'^2\Phi(\mathbf{x}')}^{-\rho(\mathbf{x}')/\varepsilon_0}]d^3x'$$

$$= \oint_S [\Phi(\mathbf{x}')\frac{\partial}{\partial n'}G_D(\mathbf{x},\mathbf{x}') - \underbrace{G_D(\mathbf{x},\mathbf{x}')}_{=\,0 \text{ on } S}\frac{\partial}{\partial n'}\Phi(\mathbf{x}')]da'$$


Thus, we obtain

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_v \rho(\mathbf{x}')G_D(\mathbf{x},\mathbf{x}')d^3x' - \frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\frac{\partial G_D(\mathbf{x},\mathbf{x}')}{\partial n'}da' \quad (1.44)$$

(1.44) expresses the solution $\Phi$ of the general electrostatic problem in (10) in terms of the solution $G_D(\mathbf{x},\mathbf{x}')$ of the point source problem in (11) and the boundary value ($\Phi_s$) of $\Phi$ on $S$. To evaluate (1.44), we first solve (11) for $G_D(\mathbf{x},\mathbf{x}')$, then substitute $G_D(\mathbf{x},\mathbf{x}')$, $\rho(\mathbf{x}')$, $\Phi_s$ into (1.44). It is often simpler to solve $G_D(\mathbf{x},\mathbf{x}')$ from (11) than solving $\Phi$ directly from (10), because (11) has the simple b.c. of $G_D(\mathbf{x},\mathbf{x}') = 0$ on $S$. Applications of (1.44) can be found in Chs. 2 and 3. The problem below gives an application without the need to solve (11) for $G(\mathbf{x},\mathbf{x}')$.

# 6.4 Green's Function for the Wave Equation

(6.15) and (6.16) have the basic form:

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = -4\pi f(\mathbf{x},t) \tag{6.32}$$

in *free space*. We assume the space is unbounded (infinite) and solve (6.32) by the Green function method. We first obtain the Green function from

See next page

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{x},t,\mathbf{x}',t') = -4\pi\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') \tag{6.41}$$

For a point source in an unbounded and isotropic medium, it is convenient to transform the origins of space and time to the source point at $\mathbf{x}'$ and $t'$, so that $G$ depends only upon $R$ and $\tau$. $\Rightarrow$

$$\begin{cases} \nabla^2 G = \dfrac{1}{R}\dfrac{\partial^2}{\partial R^2}(RG) \\ \dfrac{\partial^2}{\partial t^2}G = \dfrac{\partial^2}{\partial \tau^2}G \\ G(\mathbf{x},t,\mathbf{x}',t') = G(R,\tau) \end{cases}$$

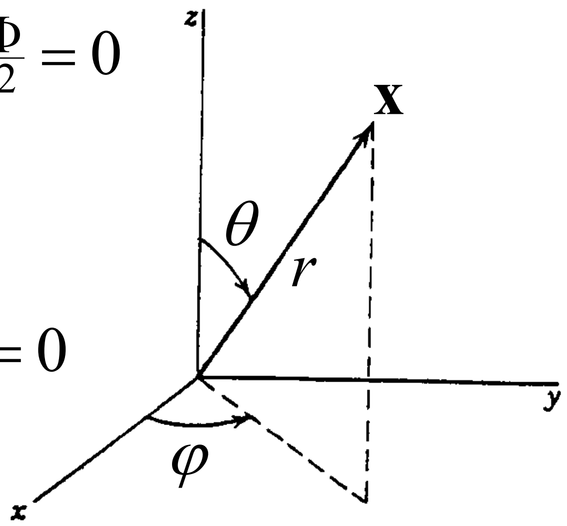
where $R = |\mathbf{x}-\mathbf{x}'|$, $\tau = t - t'$, and $\mathbf{R} = \mathbf{x}-\mathbf{x}'$. Thus, (6.41) gives

$$\frac{1}{R}\frac{\partial^2}{\partial R^2}RG(R,\tau) - \frac{1}{c^2}\frac{\partial^2}{\partial \tau^2}G(R,\tau) = -4\pi\delta(\mathbf{R})\delta(\tau) \tag{2}$$

# 3.1 Laplace Equation in Spherical Coordinates

$$\nabla^2\Phi(\mathbf{x})=0$$

$$\Rightarrow \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}=0$$




$$\text{Let } \Phi(\mathrm{x})=\frac{U(r)}{r}P(\theta)Q(\varphi)$$

$$\Rightarrow PQ\frac{d^2U}{dr^2}+\frac{UQ}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\frac{UP}{r^2\sin^2\theta}\frac{d^2Q}{d\varphi^2}=0$$

Multiply by $\dfrac{r^2\sin^2\theta}{UPQ}$

Dividing all terms by sin$^2q$, we see that the $r$-dependence is isolated within this term. So this term must be a constant. Let it be $\nu(\nu+1)$.

$$\Rightarrow \sin^2\theta\,[\overbrace{\frac{1}{U}r^2\frac{d^2U}{dr^2}}^{=\nu(\nu+1)}+\frac{1}{P\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{dP}{d\theta})]+\underbrace{\frac{1}{Q}\frac{d^2Q}{d\varphi^2}}_{=-m^2}=0 \qquad (3.3)$$

The $j$-dependence is isolated within this term, so this term must be a constant. Let it be $-m^2$.

Rewrite (2): $\frac{1}{R}\frac{\partial^2}{\partial R^2}RG(R,\tau)-\frac{1}{c^2}\frac{\partial^2}{\partial\tau^2}G(R,\tau)=-4\pi\delta(\mathbf{R})\delta(\tau)$

Performing a Fourier transform in $\tau$, we obtain

$$\frac{1}{R}\frac{d^2}{dR^2}[RG(R,\omega)]+\frac{\omega^2}{c^2}G(R,\omega)=-4\pi\delta(\mathbf{R}), \tag{6.37}$$

where
$$\begin{cases} G(R,\omega)=\int_{-\infty}^{\infty}G(R,\tau)e^{i\omega\tau}d\tau & (3) \\ G(R,\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}G(R,\omega)e^{-i\omega\tau}d\omega & (4) \end{cases}$$

In the limit $\frac{\omega}{c}\to 0$, (6.37) takes the form of the Poisson equation with a point source at $R=0$. Hence, $\boxed{\frac{1}{R}\frac{d^2}{dR^2}[RG(R,\omega)]=-4\pi\delta(\mathbf{R})}$

$$\lim_{\frac{\omega}{c}\to 0}G(R,\omega)=\frac{1}{R} \tag{6.38}$$

*Note*: Jackson defines $k\equiv\omega/c$ (p. 243) and denotes $G(R,\omega)$ by $G_k(R)$ (p. 244). Here, we retain the notation $\omega$ as an explicit reminder that $G(R,\omega)$ is an $\omega$-space quantity.

For $R>0$, (6.37) reduces to: $\frac{1}{R}\frac{d^2}{dR^2}\left[RG(R,\omega)\right]+\frac{\omega^2}{c^2}G(R,\omega)=0.$

$$\boxed{\frac{d^2}{dR^2}\left[RG(R,\omega)\right]+\frac{\omega^2}{c^2}RG(R,\omega)=0}$$

$$\Rightarrow G(R,\omega)=A\frac{e^{i\frac{\omega}{c}R}}{R}+B\frac{e^{-i\frac{\omega}{c}R}}{R} \qquad (5)$$

If $A+B=1$, (5) is also a valid solution for $R=0$ since it reduces to $\frac{1}{R}$ as $R\to 0$ [as required by (6.38)]. Hence, for $R\geq 0$, we have

$$G(R,\omega)=AG^{+}(R,\omega)+BG^{-}(R,\omega), \qquad (6.39)$$

subject to the condition $A+B=1$. In (6.39), $G^{\pm}(R,\omega)\equiv\frac{e^{\pm i\frac{\omega}{c}R}}{R}$ (6.40)

$$\Rightarrow G^{\pm}(R,\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}G^{\pm}(R,\omega)e^{-i\omega\tau}d\omega \quad \text{[from (4)]}$$

$$=\frac{1}{2\pi R}\int_{-\infty}^{\infty}e^{-i\omega(\tau\mp\frac{R}{c})}d\omega=\frac{\delta(\tau\mp\frac{R}{c})}{R} \qquad (6.43)$$

Substituting $|\mathbf{x}-\mathbf{x}'|$ for $R$ and $t-t'$ for $\tau$ into (6.43), we obtain

$$G^{\pm}(\mathbf{x},t;\mathbf{x}',t')=\frac{\delta[t'-(t\mp\frac{|\mathbf{x}-\mathbf{x}'|}{c})]}{|\mathbf{x}-\mathbf{x}'|}\quad\begin{bmatrix}G^{+}:\text{retarded Green function}\\ G^{-}:\text{advanced Green function}\end{bmatrix} \qquad (6.44)$$

We have obtained 2 solutions:

$$G^{\pm}(\mathbf{x},t;\mathbf{x}',t') = \frac{\delta\left[t' - \left(t \mp \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\right]}{|\mathbf{x}-\mathbf{x}'|} \tag{6.44}$$

for the equation:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{x},t,\mathbf{x}',t') = -4\pi\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') \tag{6.41}$$

The solution $G^+$ indicates that an effect observed at ($\mathbf{x}$, $t$) is caused by the action of a point source a distance $|\mathbf{x}-\mathbf{x}'|$ away at an *earlier* time $t' = t - |\mathbf{x}-\mathbf{x}'|/c$. This is a physical solution because the time of the cause ($t'$) precedes the time of the effect ($t$). For the $G^-$ solution, however, the time of the cause ($t' = t + |\mathbf{x}-\mathbf{x}'|/c$) would be *after* the time of the effect ($t$). This is not physically possible. Thus, "causality" requires that we reject the $G^-$ solution and set $A = 1$, $B = 0$ in (5) or (6.39). Then, the physical solution of (6.41) is $G = G^+ = \dfrac{\delta\left[t' - \left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\right]}{|\mathbf{x}-\mathbf{x}'|}$.

Going back to the basic form of (6.15) and (6.16):

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = -4\pi f(\mathbf{x},t) \qquad (6.32)$$

This equation has a distributed source $f(\mathbf{x},t)$. Since we already have the solution $G^+$ for a point source at $(\mathbf{x}',t')$, the solution for $\psi$ in (6.32) is, by the principle of linear superposition,

$$\psi(\mathbf{x},t) = \int d^3x' \int dt' \underbrace{G^+(\mathbf{x},t,\mathbf{x}',t')}_{=\delta[t'-(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c})]/|\mathbf{x}-\mathbf{x}'|} f(\mathbf{x}',t') = \int \frac{[f(\mathbf{x}',t')]_{ret}}{|\mathbf{x}-\mathbf{x}'|} d^3x' \qquad (6.47)$$

where the notation $[\ \ ]_{ret}$ implies that quantities in the brackets (including the position vector $\mathbf{x}'$) are to be evaluated at the retarded time: $t' = t - |\mathbf{x}-\mathbf{x}'|/c$. We can verify that (6.47) is the solution by sub. $\psi(\mathbf{x},t) = \int\int G^+(\mathbf{x},t,\mathbf{x}',t') f(\mathbf{x}',t') d^3x' dt'$ into (6.32) and use (6.41).

[see M&W, pp. 278-280 for an alternative derivation of (6.47)]

*Discussion*:

(i) Rewrite (6.47):

$$\psi(\mathbf{x},t)=\int\frac{[f(\mathbf{x}',t')]_{ret}}{|\mathbf{x}-\mathbf{x}'|}d^3x' \tag{6.47}$$

(6.47) is valid for unbounded space (see p. 244, bottom). If there are boundary surfaces, boundary conditions must be considered in order to account for sources on the boundary. A similar situation can be found in electrostatics, where the solution

$$\Phi(\mathbf{x})=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \tag{1.23}$$

is valid for unbounded space, while the solution

$$\Phi(\mathbf{x})=\frac{1}{4\pi\varepsilon_0}\int_V\rho(\mathbf{x}')G_D(\mathbf{x},\mathbf{x}')d^3x'-\frac{1}{4\pi}\oint_S\Phi(\mathbf{x}')\frac{\partial}{\partial n'}G_D(\mathbf{x},\mathbf{x}')da' \tag{1.44}$$

applies to a finite volume with boundary effects accounted for by the second term on the RHS.

(ii) Rewrite the Green function: $G^+ = \delta[t' - (t - \frac{|\mathbf{x}-\mathbf{x}'|}{c})]/|\mathbf{x}-\mathbf{x}'|$

This is the signal observed at $(\mathbf{x}, t)$ due to the action of a delta function source at $(\mathbf{x}', t')$. Such a source has equal components in all frequencies. If the medium is dispersive (i.e. wave speed varys with the frequency), components of the signal will propagate at different speeds and reach $\mathbf{x}$ at different times. Thus, the signal observed at $\mathbf{x}$ will be a pulse of finite duration, rather than a delta function of time as in $G^+$. This explains why the solution for $G^+$ is valid only for the free space or a non-dispersive medium [see p. 243 (top) and p. 245] in which all the wave components propagate toward $\mathbf{x}$ at the same speed and consequently reach $\mathbf{x}$ at the same instant of time.

(iii) The relation between observer's time and the retarded time, $t' = t - |\mathbf{x} - \mathbf{r}(t')|/c$, indicates that a signal from the charge travels at speed $c$ toward the observer, independent of the motion of the charge (Einstein's postulate 2).

(iv) The solution in (47): $\psi(\mathbf{x},t) = \int \frac{[f(\mathbf{x}',t')]_{ret}}{|\mathbf{x}-\mathbf{x}'|} d^3x'$ (47)

is due to the source $f$. More generally, we may add to this solution a complementary function $\psi_{in}(\mathbf{x},t)$, which is any solution of the homogeneous wave equation:

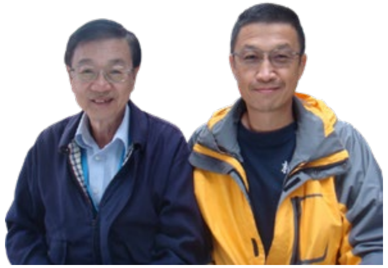
$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = 0$$

Thus, in general, the solution of $\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = -4\pi f(\mathbf{x},t)$

can be written $\psi(\mathbf{x},t) = \psi_{in}(\mathbf{x},t) + \int \frac{[f(\mathbf{x}',t')]_{ret}}{|\mathbf{x}-\mathbf{x}'|} d^3x'$ (6.45)

For example, $\psi_{in}(\mathbf{x},t)$ can be a plane wave incident on a dielectric object while $\int \frac{[f(\mathbf{x}',t')]_{ret}}{|\mathbf{x}-\mathbf{x}'|} d^3x'$ is the wave generated by the induced currents and charges in the dielectric object (treated in Ch. 9).

# 6.5 Retarded Solution for the Fields…



$$\text{Rewrite}\begin{cases}\nabla^2\Phi-\dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}\Phi=-\rho/\varepsilon_0 & (6.15)\\[2ex] \nabla^2\mathbf{A}-\dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}\mathbf{A}=-\mu_0\mathbf{J} & (6.16)\end{cases}$$

Each Cartesian component of (6.15) and (6.16) is in the form of (6.32). Assuming free space and superposing the Green function $G^+$ from all points in the distributed sources $\rho$ and $\mathbf{J}$, we obtain

$$\begin{Bmatrix}\Phi(\mathbf{x},t)\\ \mathbf{A}(\mathbf{x},t)\end{Bmatrix}=\frac{1}{4\pi}\int\!\!\int\frac{\delta\left[t'-\left(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\right]}{|\mathbf{x}-\mathbf{x}'|}\begin{Bmatrix}\rho(\mathbf{x}',t')/\varepsilon_0\\ \mu_0\mathbf{J}(\mathbf{x}',t')\end{Bmatrix}d^3x'dt'$$

$$=\frac{1}{4\pi}\int\frac{1}{R}\begin{Bmatrix}\rho(\mathbf{x}',t')/\varepsilon_0\\ \mu_0\mathbf{J}(\mathbf{x}',t')\end{Bmatrix}_{ret}d^3x',\quad R=|\mathbf{x}-\mathbf{x}'| \qquad (6.48)$$

*Note*: $\Phi$ and $\mathbf{A}$ reduce to (1.17) and (5.32), respectively, in the static limit, i.e. when $\rho$ and $\mathbf{J}$ are independent of time.

The fields **E** and **B** can be expressed in terms of $\Phi$ and **A**. We may also express **E** and **B** directly in terms of $\rho$, **J** by converting the Maxwell equations into equations for **E** and **B** in the form of (6.32).

$$\begin{cases} \nabla\cdot\mathbf{E}=\rho/\varepsilon_0 \\ \nabla\cdot\mathbf{B}=0 \\ \nabla\times\mathbf{E}=-\frac{\partial}{\partial t}\mathbf{B} \\ \nabla\times\mathbf{B}=\mu_0\mathbf{J}+\mu_0\varepsilon_0\frac{\partial}{\partial t}\mathbf{E}=\mu_0\mathbf{J}+\frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E} \end{cases} \quad \begin{bmatrix}\text{Maxwell equations} \\ \text{in free space}\end{bmatrix}$$

$$\nabla\times\nabla\times\mathbf{E}=-\frac{\partial}{\partial t}\nabla\times\mathbf{B} \quad \Rightarrow \quad \nabla\underbrace{(\nabla\cdot\mathbf{E})}_{\rho/\varepsilon_0}-\nabla^2\mathbf{E}=-\mu_0\frac{\partial}{\partial t}\mathbf{J}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}$$

$$\Rightarrow \nabla^2\mathbf{E}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}=\frac{1}{\varepsilon_0}\nabla\rho+\mu_0\frac{\partial}{\partial t}\mathbf{J}=-\frac{1}{\varepsilon_0}\left(-\nabla\rho-\frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{J}\right) \qquad (6.49)$$

$$\nabla\times\nabla\times\mathbf{B}=\mu_0\nabla\times\mathbf{J}+\frac{1}{c^2}\frac{\partial}{\partial t}\nabla\times\mathbf{E}$$

$$\Rightarrow \nabla(\nabla\cdot\mathbf{B})-\nabla^2\mathbf{B}=\mu_0\nabla\times\mathbf{J}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{B}$$

$$\Rightarrow \nabla^2\mathbf{B}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{B}=-\mu_0\nabla\times\mathbf{J} \qquad (6.50)$$

(6.49) and (6.50) are in the same form as (6.32). Assuming infinite space and apply the Green function $G^+$, we obtain

$$\mathbf{E}(\mathbf{x},t)=\frac{1}{4\pi\varepsilon_0}\iint\frac{\delta\left[t'-\left(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\right]\left[-\nabla'\rho-\frac{1}{c^2}\frac{\partial\mathbf{J}}{\partial t'}\right]}{|\mathbf{x}-\mathbf{x}'|}d^3x'dt'$$

$$=\frac{1}{4\pi\varepsilon_0}\int\frac{1}{R}\left[-\nabla'\rho-\frac{1}{c^2}\frac{\partial\mathbf{J}}{\partial t'}\right]_{ret}d^3x' \qquad (6.51)$$

$$\mathbf{B}(\mathbf{x},t)=\frac{\mu_0}{4\pi}\iint\frac{\delta\left[t'-\left(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\right]\nabla'\times\mathbf{J}}{|\mathbf{x}-\mathbf{x}'|}d^3x'dt'$$

$$=\frac{\mu_0}{4\pi}\int\frac{1}{R}[\nabla'\times\mathbf{J}]_{ret}d^3x' \qquad (6.52)$$

(6.51) and (6.52) can be converted into the Jefimenko formulae [see (6.55) and (6.56)], which explicitly show the reduction to the static equations (1.5) and (5.14).

Griffiths

Optional

# 10.2.2 Jefimenko's Equations

Retarded potentials:

$$V(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}}d\tau' \quad \text{and} \quad \mathbf{A}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathscr{r}}d\tau'$$

$$\mathbf{E}=-\nabla V-\frac{\partial\mathbf{A}}{\partial t} \quad \begin{cases} -\nabla V=\dfrac{1}{4\pi\varepsilon_0}\displaystyle\int[\frac{\dot{\rho}\hat{\mathscr{r}}}{c\mathscr{r}}+\frac{\rho\hat{\mathscr{r}}}{\mathscr{r}^2}]d\tau' \\ -\dfrac{\partial\mathbf{A}}{\partial t}=-\dfrac{\partial}{\partial t_r}(\dfrac{\mu_0}{4\pi}\displaystyle\int\frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathscr{r}}d\tau')\frac{\partial t_r}{\partial t}=-\frac{\mu_0}{4\pi}\int\frac{\dot{\mathbf{J}}}{\mathscr{r}}d\tau' \end{cases}$$

$$\mathbf{E}=\frac{1}{4\pi\varepsilon_0}\int[\frac{\dot{\rho}\hat{\mathscr{r}}}{c\mathscr{r}}+\frac{\rho\hat{\mathscr{r}}}{\mathscr{r}^2}]d\tau'-\frac{\mu_0}{4\pi}\int\frac{\dot{\mathbf{J}}}{\mathscr{r}}d\tau'$$

$$=\frac{1}{4\pi\varepsilon_0}\int[\frac{\rho\hat{\mathscr{r}}}{\mathscr{r}^2}+\frac{\dot{\rho}\hat{\mathscr{r}}}{c\mathscr{r}}-\frac{\dot{\mathbf{J}}}{c^2\mathscr{r}}]d\tau'$$

The time-dependent generalization of Coulomb's law.

Griffiths

Optional

# Jefimenko's Equations (ii)

Retarded potentials:

$$V(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}}d\tau' \quad \text{and} \quad \mathbf{A}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathscr{r}}d\tau'$$

$$\mathbf{B}=\nabla\times\mathbf{A}=\frac{\mu_0}{4\pi}\int\nabla\times\frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathscr{r}}d\tau'=\frac{\mu_0}{4\pi}\int[\frac{1}{\mathscr{r}}\nabla\times\mathbf{J}-\mathbf{J}\times\nabla\frac{1}{\mathscr{r}}]d\tau'$$

$$\nabla\times\mathbf{J}=\frac{1}{c}\dot{\mathbf{J}}\times\hat{\mathscr{r}} \quad \text{and} \quad \nabla(\frac{1}{\mathscr{r}})=-\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\mathbf{B}=\frac{\mu_0}{4\pi}\int[\frac{\mathbf{J}}{\mathscr{r}^2}+\frac{1}{c\mathscr{r}}\dot{\mathbf{J}}]\times\hat{\mathscr{r}}d\tau'$$

The time-dependent generalization of the Biot-Savart law.

These two equations are *of limited utility*, but they provide a satisfying sense of closure to the theory.

## 6.7 Poynting's Theorem and Conservation of Energy and Momentum for a System of Particles and Electromagnetic Fields

$$\boxed{dw = \mathbf{f}\cdot d\vec{\ell},\quad \frac{dw}{dt} = \mathbf{f}\cdot\mathbf{v},\quad \frac{dW}{dt} = \int \frac{dw}{dt} d^3x = \int \mathbf{f}\cdot\mathbf{v} d^3x}$$

The rate of work done by the **E**-field on charged particles inside a volume $V$ is given by

$$\boxed{\mathbf{f} = \rho(\mathbf{E} + \mathbf{v}\times\mathbf{B})} \qquad \boxed{\mathbf{J} = \nabla\times\mathbf{H} - \frac{\partial}{\partial t}\mathbf{D}}$$

$$\int_V \mathbf{f}\cdot\mathbf{v} d^3x = \int_V \rho\mathbf{v}\cdot\mathbf{E} d^3x = \int_V \mathbf{J}\cdot\mathbf{E} d^3x = \int_V (\underbrace{\mathbf{E}\cdot\nabla\times\mathbf{H}} - \mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D}) d^3x$$

$$\boxed{= \mathbf{H}\cdot\underbrace{\nabla\times\mathbf{E}}_{-\frac{\partial}{\partial t}\mathbf{B}} - \nabla\cdot(\mathbf{E}\times\mathbf{H}) = -\mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B} - \nabla\cdot(\mathbf{E}\times\mathbf{H})}$$

$$\Rightarrow \underbrace{\int_V \mathbf{J}\cdot\mathbf{E} d^3x}_{} = -\int_V \left[\nabla\cdot(\mathbf{E}\times\mathbf{H}) + \mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D} + \mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B}\right] d^3x \qquad (6.105)$$

rate of conversion of EM energy into mechanical and thermal energies.

Rewrite (6.105): $\int_V \mathbf{J}\cdot\mathbf{E}d^3x = -\int_V \left[\nabla\cdot(\mathbf{E}\times\mathbf{H}) + \mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D} + \mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B}\right]d^3x$

The terms $\mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D}$ and $\mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B}$ in the integrand can be interpreted physically if we make the following assumptions:

*Assumption 1*: The medium is *linear* with *negligible dispersion* and *negligible losses*.

We can then write (reasons given in the lecture notes of Ch. 7)

$$\mathbf{D}(\mathbf{x},t) = \varepsilon\mathbf{E}(\mathbf{x},t),\ \mathbf{B}(\mathbf{x},t) = \mu\mathbf{H}(\mathbf{x},t)$$

$$\Rightarrow\ \mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D} = \frac{1}{2}\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{D}),\ \mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B} = \frac{1}{2}\frac{\partial}{\partial t}(\mathbf{H}\cdot\mathbf{B}). \qquad (6)$$

*Assumption 2*: The field energy density for static fields

$$u = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}) \qquad (6.106)$$

represents the field energy density even for *time-dependent* fields.

From (6) and (6.106), we have

$$\frac{\partial u}{\partial t} = \mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D} + \mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B} = \begin{bmatrix}\text{rate of change of}\\ \text{field energy density}\end{bmatrix} \qquad (7)$$

## 6.7 Poynting's Theorem … (*continued*)

Rewrite (6.105): $\int_v \mathbf{J}\cdot\mathbf{E}d^3x = -\int_v \left[\nabla\cdot(\mathbf{E}\times\mathbf{H}) + \mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D} + \mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B}\right]d^3x$

Sub. $\frac{\partial u}{\partial t}$ for $\mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D} + \mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B}$, we obtain

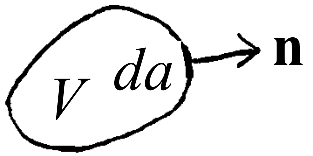
$$\int_v \mathbf{J}\cdot\mathbf{E}d^3x + \int_v \frac{\partial u}{\partial t}d^3x + \int_v \nabla\cdot(\mathbf{E}\times\mathbf{H})d^3x = 0 \qquad (6.107)$$

$$\Rightarrow \mathbf{J}\cdot\mathbf{E} + \frac{\partial u}{\partial t} + \nabla\cdot\mathbf{S} = 0 \qquad (6.108)$$

where, $\mathbf{S} \equiv \mathbf{E}\times\mathbf{H}$, is called the Poynting vector.

The meaning of $\mathbf{S}$ becomes clear if we write (6.107) as

$$\underbrace{\int_v \mathbf{J}\cdot\mathbf{E}d^3x}_{\frac{d}{dt}E_{mech}} + \underbrace{\int_v \frac{\partial u}{\partial t}d^3x}_{\frac{d}{dt}E_{field}} + \underbrace{\int_v \nabla\cdot(\mathbf{E}\times\mathbf{H})d^3x}_{\oint_s \mathbf{S}\cdot\mathbf{n}da} = 0$$



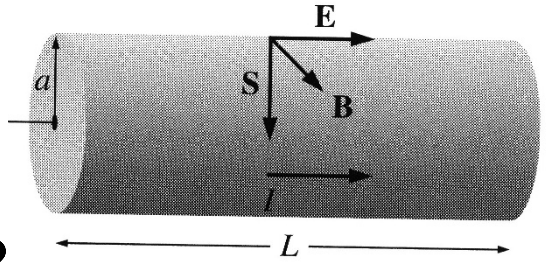

$$\Rightarrow \frac{d}{dt}\left(E_{mech} + E_{field}\right) = -\oint_s \mathbf{S}\cdot\mathbf{n}da \text{ [Poynting's theorem]} \qquad (6.111)$$

where $E_{mech}$ is the total mechanical/thermal energies inside $V$ (no particles move in or out of $V$) and $E_{field}$ the total field energy inside $V$. Then, by conservation of energy, $\mathbf{S}$ is the power/unit area.

Griffiths

# Example 8.1

When current flows down a wire, work is done, which shows up as Joule heating of the wire. Find the energy per unit time delivered to the wire using Poynting vector?




Sol:

$$\mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) \qquad \begin{cases} \mathbf{E} = \dfrac{V}{L}\hat{\mathbf{z}} \\ \mathbf{B}(r=a) = \dfrac{\mu_0 I}{2\pi a}\hat{\boldsymbol{\phi}} \end{cases}$$

So $$\mathbf{S} = \frac{1}{\mu_0}\left(\frac{V}{L}\hat{\mathbf{z}} \times \frac{\mu_0 I}{2\pi a}\hat{\boldsymbol{\phi}}\right) = -\frac{VI}{2\pi a L}\hat{\mathbf{r}}$$ (point radially inward)

The energy per unit time passing through the surface of the wire is:

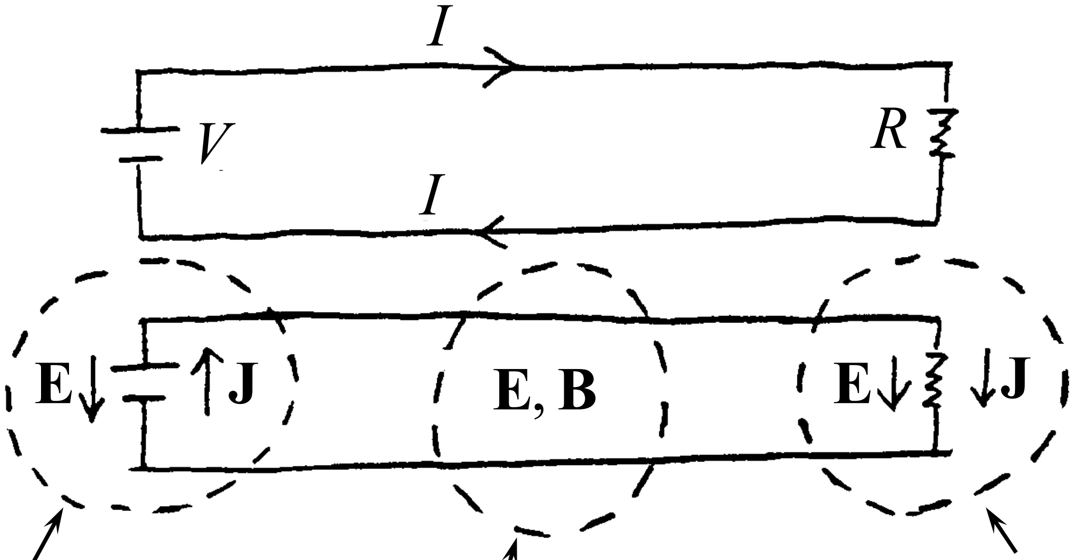
$$-\oint_S \mathbf{S}\cdot d\mathbf{a} = S(2\pi a L) = VI = \frac{dW}{dt} \qquad \frac{dU_{\text{em}}}{dt} = 0 \text{ (static fields)}$$

*Example*: a DC circuit

$$\int_V \mathbf{J}\cdot\mathbf{E}\,d^3x + \int_V \frac{\partial u}{\partial t}\,d^3x + \oint_S \mathbf{S}\cdot\mathbf{n}\,da = 0 \;\underset{\text{steady state}}{\Rightarrow}\; \int_V \mathbf{J}\cdot\mathbf{E}\,d^3x + \oint_S \mathbf{S}\cdot\mathbf{n}\,da = 0$$

circuit view:

field view:




$\int_V \mathbf{J}\cdot\mathbf{E}\,d^3x < 0$

$\Rightarrow \oint_S \mathbf{S}\cdot\mathbf{n}\,da > 0$

$\Rightarrow$ Power flows out of battery.

Power transmission by Poynting vector

$\int_V \mathbf{J}\cdot\mathbf{E}\,d^3x > 0$

$\Rightarrow \oint_S \mathbf{S}\cdot\mathbf{n}\,da < 0$

$\Rightarrow$ Power flows into resistor.

*Give a man a fish and you feed him for a day;*
*teach a man to fish and you feed him for a lifetime.*

## Conservation of Linear Momentum of a System of Particles and Fields:

Write down the Maxwell equations in the vacuum medium:

$$\begin{cases} \rho = \varepsilon_0 \nabla \cdot \mathbf{E} \\ \mathbf{J} = \dfrac{1}{\mu_0} \nabla \times \mathbf{B} - \varepsilon_0 \dfrac{\partial}{\partial t} \mathbf{E} \end{cases}$$

$$\boxed{\begin{aligned} \mathbf{B} \times \frac{\partial}{\partial t} \mathbf{E} &= -\frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B}) + \mathbf{E} \times \frac{\partial}{\partial t} \mathbf{B} \\ &= -\frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B}) - \mathbf{E} \times (\nabla \times \mathbf{E}) \end{aligned}}$$

$$\Rightarrow \mathbf{f} = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B} = \varepsilon_0 [\mathbf{E}(\nabla \cdot \mathbf{E}) + \overbrace{\mathbf{B} \times \frac{\partial}{\partial t} \mathbf{E}} - c^2 \mathbf{B} \times (\nabla \times \mathbf{B})]$$

$$= \varepsilon_0 [\mathbf{E}(\nabla \cdot \mathbf{E}) - \mathbf{E} \times (\nabla \times \mathbf{E}) + \underbrace{c^2 \mathbf{B}(\nabla \cdot \mathbf{B})} - c^2 \mathbf{B} \times (\nabla \times \mathbf{B})] - \varepsilon_0 \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B})$$

This term, which equals 0, is added for later manipulation.

Sub. the expression for the force density $\mathbf{f}$ into Newton's 2nd law:

$$\frac{d}{dt} \mathbf{P}_{mech} = \int_V \mathbf{f} d^3 x \quad [\mathbf{P}_{mech} : \text{total momentum of all particles in } V.]$$

we obtain $\dfrac{d}{dt} \mathbf{P}_{mech} + \dfrac{d}{dt} \displaystyle\int_V \varepsilon_0 (\mathbf{E} \times \mathbf{B}) d^3 x$

$$= \varepsilon_0 \int_V \left[ \mathbf{E}(\nabla \cdot \mathbf{E}) - \mathbf{E} \times (\nabla \times \mathbf{E}) + c^2 \mathbf{B}(\nabla \cdot \mathbf{B}) - c^2 \mathbf{B} \times (\nabla \times \mathbf{B}) \right] d^3 x \quad (6.116)$$

Rewrite (6.116):

$$\frac{d}{dt}\mathbf{P}_{mech}+\frac{d}{dt}\int_V \varepsilon_0(\mathbf{E}\times\overbrace{\mathbf{B}}^{\mu_0\mathbf{H}})d^3x$$

$$=\varepsilon_0\int_V[\mathbf{E}(\nabla\cdot\mathbf{E})-\mathbf{E}\times(\nabla\times\mathbf{E})+c^2\mathbf{B}(\nabla\cdot\mathbf{B})-c^2\mathbf{B}\times(\nabla\times\mathbf{B})]d^3x$$

Define $\mathbf{g}\equiv\frac{1}{c^2}\mathbf{E}\times\mathbf{H}$ [<u>electromagnetic momentum density</u>] (6.118)

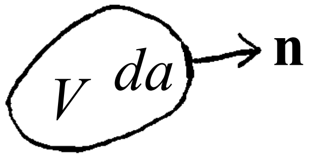
$\Rightarrow \mathbf{P}_{field}=\int_V \mathbf{g}d^3x$ [total electromagnetic momentum in $V$]

(6.116) can then be written (see p.261)

by divergence thm.

$$\frac{d}{dt}(\mathbf{P}_{mech}+\mathbf{P}_{field})_\alpha=\sum_\beta\int_V\frac{\partial}{\partial x_\beta}T_{\alpha\beta}d^3x=\oint_S\sum_\beta T_{\alpha\beta}n_\beta da \qquad (6.122)$$

$$\Rightarrow \frac{d}{dt}(\mathbf{P}_{mech}+\mathbf{P}_{field})=\oint_S\overleftrightarrow{\mathbf{T}}\cdot\mathbf{n}da \qquad (8)$$




where $\overleftrightarrow{\mathbf{T}}=\left[T_{\alpha\beta}\right]$ is the <u>Maxwell stress tensor</u> defined as

$$T_{\alpha\beta}\equiv\varepsilon_0\left[E_\alpha E_\beta+c^2B_\alpha B_\beta-\tfrac{1}{2}(\mathbf{E}\cdot\mathbf{E}+c^2\mathbf{B}\cdot\mathbf{B})\delta_{\alpha\beta}\right] \qquad (6.120)$$

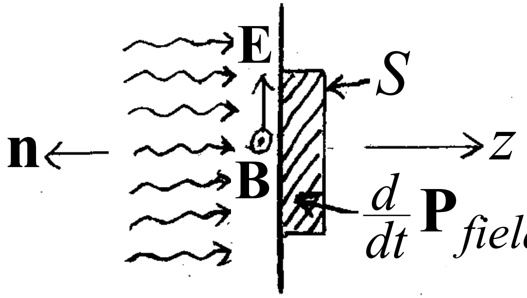
*Note*: By Newton's law, only $\frac{d}{dt}\mathbf{P}_{mech}$ (not $\frac{d}{dt}\mathbf{P}_{field}$) is the force on $V$.

**Example 1**: A plane wave is incident normally from free space onto a flat surface and is totally absorbed. Find the force on the surface.

*Solution*: Consider the volume enclosed by $S$. On the left side, we have

$$\mathbf{n} = -\mathbf{e}_z = (0,0,-1)$$

$$\left.\begin{array}{l}\mathbf{E} = (E_x, E_y, 0)\\ \mathbf{B} = (B_x, B_y, 0)\end{array}\right\} \left[\begin{array}{l}\text{instantaneous}\\ \text{fields on the}\\ \text{left surface}\end{array}\right]$$




$$\frac{d}{dt}\mathbf{P}_{field} = 0$$

$$\overleftrightarrow{\mathbf{T}} \cdot \mathbf{n} = \varepsilon_0 \begin{bmatrix} E_x^2 + c^2 B_x^2 - \frac{1}{2}(E^2 + c^2 B^2) & E_x E_y + c^2 B_x B_y & 0 \\ E_y E_x + c^2 B_y B_x & E_y^2 + c^2 B_y^2 - \frac{1}{2}(E^2 + c^2 B^2) & 0 \\ 0 & 0 & -\frac{1}{2}(E^2 + c^2 B^2) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$= \frac{1}{2}\varepsilon_0 (E^2 + c^2 B^2)\mathbf{e}_z = \frac{1}{2}(\varepsilon_0 E^2 + \frac{B^2}{\mu_0})\mathbf{e}_z \quad [c^2 = \frac{1}{\mu_0 \varepsilon_0}]$$

area of left surface

$$\frac{d}{dt}(\mathbf{P}_{mech} + \mathbf{P}_{field}) = \oint_S \overleftrightarrow{\mathbf{T}} \cdot \mathbf{n}\, da \Rightarrow \mathbf{F} = \frac{d}{dt}\mathbf{P}_{mech} = \frac{1}{2}(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}) A \mathbf{e}_z$$

$$\Rightarrow \left[\begin{array}{l}\text{instantaneous}\\ \text{radiation pressure}\end{array}\right] = \frac{\mathbf{F}}{A} = \underbrace{\frac{1}{2}(\varepsilon_0 E^2 + \frac{B^2}{\mu_0})}_{\text{instantaneous energy density}} \mathbf{e}_z$$

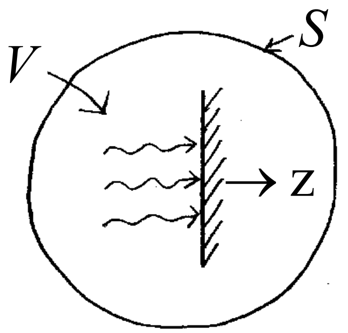
*Alternative solution to* **Example 1**:

Assume the plane wave has a finite cross section $A$ and a finite length. We may then enclose the full extent of the wave within surface $S$ (see figure). There is no field on the surface. Hence, for volume $V$,

electromagnetic momentum density

$$\frac{d}{dt}(\mathbf{P}_{mech}+\mathbf{P}_{field})=\oint_s \overleftrightarrow{\mathbf{T}}\cdot\mathbf{n}da=0$$

$$\Rightarrow \mathbf{F}=\frac{d}{dt}\mathbf{P}_{mech}=-\frac{d}{dt}\mathbf{P}_{field}=-\frac{d}{dt}\int_v \mathbf{g}d^3x,$$




where $\mathbf{g}=\frac{1}{c^2}\mathbf{E}\times\mathbf{H}=\frac{1}{c^2}P\mathbf{e}_z$ [by (6.118) and (6.109)]

Because the wave travels at speed $c$ and it is totally absorbed, the electromagnetic momentum $\mathbf{P}_{field}$ in $V$ decreases at the rate $\mathbf{g}cA$.

$$\Rightarrow \begin{cases} \mathbf{F}=\frac{1}{c}PA\mathbf{e}_z \\ \frac{\mathbf{F}}{A}=\frac{1}{c}P\mathbf{e}_z=\frac{1}{2}(\varepsilon_0 E^2+\frac{1}{\mu_0}B^2)\mathbf{e}_z \end{cases}$$

*Note*: This method does not require the absorbing material to be flat.

*Question*: The radiation pressure is due to the $\mathbf{J}\times\mathbf{B}$ force. How?

**Example 2**: A spherical particle in the outer space with radius *r*, mass *M*, and density $\rho_M = 3.5\times10^3$ kg/m$^3$ absorbs all the sunlight it intercepts. For what value of *r* does the sun's radiation force ($F_R$) on the particle balance the sun's gravitational force ($F_G$).

time-averaged radiation pressure (see Ex. 1) = $I / c$

*I*: sunlight intensity (average power/unit area) at the particle
$P_S$: total power radiated by sun ($3.9\times10^{26}$ W)
*R*: distance to sun

*Solution*:

$$F_R = \overbrace{\frac{1}{2}\left\langle \varepsilon_0 E^2 + \frac{B^2}{\mu_0} \right\rangle_t}^{I/c} \pi r^2 = \frac{I\pi r^2}{c} = \frac{P_S \pi r^2}{4\pi R^2 c}$$

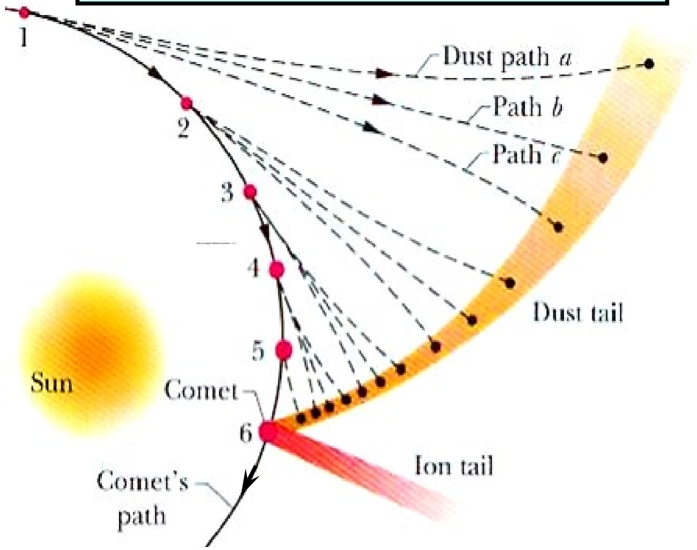
*G*: gravitational const. ($6.67\times10^{-11}$ Nm$^2$/kg$^2$)
$M_S$: sun's mass ($1.99\times10^{30}$ kg)

$$F_G = \frac{GM_S M}{R^2} = \frac{GM_S}{R^2}\frac{4\pi r^3 \rho_M}{3}$$

$$F_R = F_G \Rightarrow r = \frac{3P_S}{16\pi c \rho_m G M_S} = 1.7\times10^{-7}\,\text{m}$$

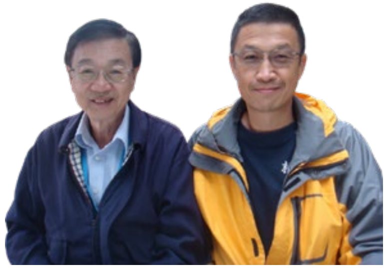
$$\Rightarrow F_G \begin{cases} > \\ = \\ < \end{cases} F_R \quad \text{if } r \begin{cases} > \\ = \\ < \end{cases} 1.7\times10^{-7}\,\text{m}$$

Size does matter.




from Haliday, Resnick, and Walker

# 6.9 Poynting's Theorem for Harmonic Fields; Field Definitions of Impedance and Admittance



**Phasors :**

In linear equations, harmonic quantities can be represented by complex variables as follows:

$$\underbrace{\begin{Bmatrix} \mathbf{E}(\mathbf{x},t) \\ \mathbf{D}(\mathbf{x},t) \\ \mathbf{B}(\mathbf{x},t) \\ \mathbf{H}(\mathbf{x},t) \\ \mathbf{J}(\mathbf{x},t) \\ \rho(\mathbf{x},t) \end{Bmatrix}}_{\text{real}} = \mathrm{Re}\left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \begin{Bmatrix} \mathbf{E}(\mathbf{x},\omega) \\ \mathbf{D}(\mathbf{x},\omega) \\ \mathbf{B}(\mathbf{x},\omega) \\ \mathbf{H}(\mathbf{x},\omega) \\ \mathbf{J}(\mathbf{x},\omega) \\ \rho(\mathbf{x},\omega) \end{Bmatrix} e^{-i\omega t} d\omega \right]$$

Fourier Transform
Eqs. 2.44 & 2.45

complex (called the phasor)

It is assumed that the LHS is given by the real part of the RHS.

## Representation of Time-Averaged Quantities by Phasors :

Assume that all fields and sources have a time dependence $e^{-i\omega t}$, so we write the quantities as

$$\mathbf{E}(\mathbf{x},t) = \mathrm{Re}[\mathbf{E}(\mathbf{x})e^{-i\omega t}] = \tfrac{1}{2}[\mathbf{E}(\mathbf{x})e^{-i\omega t} + \mathbf{E}^*(\mathbf{x})e^{i\omega t}]$$

$$\mathbf{J}(\mathbf{x},t) = \mathrm{Re}[\mathbf{J}(\mathbf{x})e^{-i\omega t}] = \tfrac{1}{2}[\mathbf{J}(\mathbf{x})e^{-i\omega t} + \mathbf{J}^*(\mathbf{x})e^{i\omega t}]$$

Then,

$$\mathbf{J}(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x},t)$$

$$= \tfrac{1}{4}[\mathbf{J}^*(\mathbf{x})\cdot\mathbf{E}(\mathbf{x}) + \mathbf{J}(\mathbf{x})\cdot\mathbf{E}^*(\mathbf{x}) + \mathbf{J}(\mathbf{x})\cdot\mathbf{E}(\mathbf{x})e^{-2i\omega t} + \mathbf{J}^*(\mathbf{x})\cdot\mathbf{E}^*(\mathbf{x})e^{2i\omega t}]$$

$$= \tfrac{1}{2}\mathrm{Re}[\mathbf{J}^*(\mathbf{x})\cdot\mathbf{E}(\mathbf{x}) + \mathbf{J}(\mathbf{x})\cdot\mathbf{E}(\mathbf{x})e^{-2i\omega t}]$$

$$= \tfrac{1}{2}\mathrm{Re}[\mathbf{J}(\mathbf{x})\cdot\mathbf{E}^*(\mathbf{x})]$$

and the time average can be written in terms of phasors as

$$\langle \mathbf{J}(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x},t)\rangle_t = \tfrac{1}{2}\mathrm{Re}[\mathbf{J}^*(\mathbf{x})\cdot\mathbf{E}(\mathbf{x})], \text{ assuming } \omega \text{ is real} \qquad (9)$$

Similary, $\langle \mathbf{E}(\mathbf{x},t)\times\mathbf{H}(\mathbf{x},t)\rangle_t = \tfrac{1}{2}\mathrm{Re}[\mathbf{E}(\mathbf{x})^*\times\mathbf{H}(\mathbf{x})]$ (10)

**Maxwell Equations in Terms of Phasors :**

In terms of phasors, the Maxwell equations can be written:

$$\begin{cases} \nabla\cdot\mathbf{B}(\mathbf{x},t)=0 \\ \nabla\times\mathbf{E}(\mathbf{x},t)=-\frac{\partial}{\partial t}\mathbf{B}(\mathbf{x},t) \\ \nabla\cdot\mathbf{D}(\mathbf{x},t)=\rho(\mathbf{x},t) \\ \nabla\times\mathbf{H}(\mathbf{x},t)=\mathbf{J}(\mathbf{x},t)+\frac{\partial}{\partial t}\mathbf{D}(\mathbf{x},t) \end{cases} \Rightarrow \begin{cases} \nabla\cdot\mathbf{B}(\mathbf{x})=0 \\ \nabla\times\mathbf{E}(\mathbf{x})=i\omega\mathbf{B}(\mathbf{x}) \\ \nabla\cdot\mathbf{D}(\mathbf{x})=\rho(\mathbf{x}) \\ \nabla\times\mathbf{H}(\mathbf{x})=\mathbf{J}(\mathbf{x})-i\omega\mathbf{D}(\mathbf{x}) \end{cases}$$

**Complex Poynting's Theorem :**

Using the phasor representation of Maxwell equations, we obtain

$$\frac{1}{2}\int_v \mathbf{J}^*\cdot\mathbf{E}d^3x=\frac{1}{2}\int_v[\overbrace{\mathbf{E}\cdot\nabla\times\mathbf{H}^*}^{-\nabla\cdot(\mathbf{E}\times\mathbf{H}^*)+\mathbf{H}^*\cdot\overbrace{\nabla\times\mathbf{E}}^{i\omega\mathbf{B}}}-i\omega\mathbf{E}\cdot\mathbf{D}^*]d^3x$$

$$=\frac{1}{2}\int_v\left[-\nabla\cdot(\mathbf{E}\times\mathbf{H}^*)-i\omega(\mathbf{E}\cdot\mathbf{D}^*-\mathbf{B}\cdot\mathbf{H}^*)\right]d^3x \quad (6.131)$$

Rewrite (6.131):

$$\frac{1}{2}\int_v \mathbf{J}^* \cdot \mathbf{E} d^3x = \frac{1}{2}\int_v [-\nabla\cdot(\mathbf{E}\times\mathbf{H}^*) - i\omega(\mathbf{E}\cdot\mathbf{D}^* - \mathbf{B}\cdot\mathbf{H}^*)] d^3x \quad (6.131)$$

This equation gives the complex Poynting theorem:

$$\frac{1}{2}\int_v \mathbf{J}^* \cdot \mathbf{E} d^3x + 2i\omega\int_v (w_e - w_m) d^3x + \oint_s \mathbf{S}\cdot\mathbf{n} da = 0 \quad (6.134)$$

where $\mathbf{S} \equiv \frac{1}{2}\mathbf{E}\times\mathbf{H}^*$ [called the complex Poynting vector] (6.132)

and the real part of $\mathbf{S}$ is the time-averaged power [see (10)].

In (6.134), $w_e$ and $w_m$ are defined as

$$\begin{cases} w_e \equiv \frac{1}{4}\mathbf{E}\cdot\mathbf{D}^* \\ w_m \equiv \frac{1}{4}\mathbf{B}\cdot\mathbf{H}^* \end{cases} \quad \left[\begin{array}{l}\text{The real part of } w_e\ (w_m) \text{ is the time} \\ \text{averaged } \mathbf{E}\ (\mathbf{B}) \text{ field energy density.}\end{array}\right] \quad (6.133)$$

If $w_e$ and $w_m$ are both real, the real part of (6.134) gives

$$\frac{1}{2}\int_v \mathrm{Re}[\mathbf{J}^* \cdot \mathbf{E}] d^3x + \oint_s \mathrm{Re}[\mathbf{S}\cdot\mathbf{n}] da = 0,$$

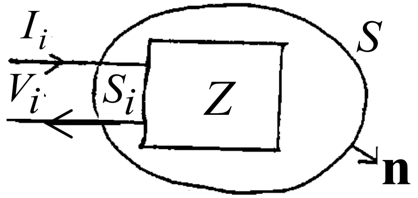
which is the counterpart of (6.107) applicable to constant-amplitude harmonic fields (for which the field energy remains constant).

Self-learning

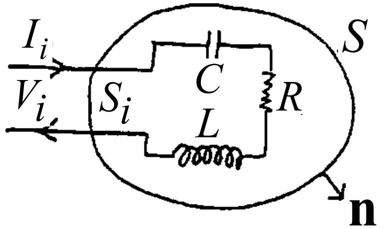


## Field Definition of Impedance :

We now apply the complex Poynting's theorem to a 2-terminal circuit. Draw a closed surface $S$ surrounding the circuit. Let $I_i$ be the input current, $V_i$ be the input voltage, and let the input energy flow be confined to a small area $S_i$. Then,




general circuit




a specific example

$$\frac{1}{2}I_i^*V_i = -\int_{S_i} \mathbf{S}\cdot\mathbf{n}da \qquad (6.135)$$

and the complex Poynting's theorem [(6.134)]

$$\frac{1}{2}\int_v \mathbf{J}^*\cdot\mathbf{E}d^3x + 2i\omega\int_v (w_e - w_m)d^3x + \oint_s \mathbf{S}\cdot\mathbf{n}da = 0$$ can be written:

$$\frac{1}{2}I_i^*V_i = \frac{1}{2}\int_v \mathbf{J}^*\cdot\mathbf{E}d^3x + 2i\omega\int_v (w_e - w_m)d^3x + \underbrace{\int_{S-S_i} \mathbf{S}\cdot\mathbf{n}da}_{\boxed{\text{radiation loss}}} = \frac{1}{2}|I_i|^2 Z,$$

where $Z$ is the impedance of the circuit defined as

$$Z \equiv \frac{V_i}{I_i} = \frac{1}{|I_i|^2}\left[\int_v \mathbf{J}^*\cdot\mathbf{E}d^3x + 4i\omega\int_v (w_e - w_m)d^3x + 2\int_{S-S_i} \mathbf{S}\cdot\mathbf{n}da\right]$$

$$= R - iX \quad (R: \text{ resistance}, X: \text{ reactance}) \qquad (6.137)\ (6.138)$$

Rewrite:

$$Z \equiv \frac{V_i}{I_i} = \frac{1}{|I_i|^2}\left[\int_v \underbrace{\mathbf{J}^* \cdot \mathbf{E}}_{=\sigma^*|E|^2} d^3x + 4i\omega \int_v (w_e - w_m) d^3x + 2\int_{S-S_i} \mathbf{S}\cdot\mathbf{n}da\right]$$

conductivity

*Special case*: $\begin{cases} \text{Assume } \mathbf{J} = \sigma\mathbf{E} \text{ and } \sigma \text{ is generally real.} \\ \text{Neglect the radiation loss term: } \int_{S-S_i} \mathbf{S}\cdot\mathbf{n}da \end{cases}$

$$\Rightarrow Z = \frac{2P - 4i\omega(W_m - W_e)}{|I_i|^2}$$

[A general definition of the impedance of a circuit in terms of the power loss and the field energy in the circuit]

where $\begin{cases} P = \frac{1}{2}\int \sigma^* |E|^2 d^3x \quad \text{[Ohmic loss]} \\ W_m = \int w_m d^3x \quad [\mathbf{B}\text{-field energy}] \\ W_e = \int w_e d^3x \quad [\mathbf{E}\text{-field energy}] \end{cases}$

and $W_m > W_e \Rightarrow$ positive reactance; $W_m < W_e \Rightarrow$ negative reactance.

This expression for $Z$ is useful for microwave circuit studies.

# Homework of Chap. 6

**Problem 6.8**

A dielectric sphere of dielectric constant ε and radius $a$ is located at the origin. There is a uniform applied electric field $E_0$ in the $x$ direction. The sphere rotates with an angular velocity ω about the $z$ axis. Show that there is a magnetic field $\mathbf{H} = -\nabla\Phi_M$, where

$$\Phi_M = \frac{3}{5}\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)\varepsilon_0 E_0\omega\left(\frac{a}{r_>}\right)^5 \cdot xz$$

Where $r_>$ is the larger of $r$ and $a$. The motion is nonrelativistic.

You may use the results of Section 4.4 for the dielectric sphere in an applied field.

**Problem 6.10**

With the same assumptions as in Problem 6.9 discuss the conservation of angular momentum. Show that the differential and integral forms of the conservation law are

$$\frac{\partial}{\partial t}\left(\mathcal{L}_{mech} + \mathcal{L}_{field}\right) + \nabla\cdot\overleftrightarrow{\mathbf{M}} = 0$$

and

$$\frac{d}{dt}\int_V\left(\mathcal{L}_{mech} + \mathcal{L}_{field}\right)d^3x + \int_S \mathrm{n}\cdot\overleftrightarrow{\mathbf{M}}da = 0$$

Where the field angular-momentum density is

$$\mathcal{L}_{field} = \mathbf{x}\times\mathbf{g} = \mu\varepsilon\mathbf{x}\times\left(\mathbf{E}\times\mathbf{H}\right)$$

and the flux of angular momentum is described by the tensor

$$\overleftrightarrow{\mathbf{M}} = \overleftrightarrow{\mathbf{T}}\times\mathbf{x}$$

Note: Here we have used the dyadic notation for $M_{ij}$ and $T_{ij}$. A double-headed arrow conveys a fairly obvious meaning. For example, $\mathbf{n}\cdot\overleftrightarrow{\mathbf{M}}$ is a vector whose $j$th component is $\sum_i n_i M_{ij}$. The second-rank $\overleftrightarrow{\mathbf{M}}$ can be written as a third-rank tensor, $M_{ijk} = T_{ij}x_k - T_{ik}x_j$. But in the indices $j$ and $k$ it is antisymmetric and so has only three independent elements. Including the index $i$, $M_{ijk}$ therefore has nine components and can be written as a pseudotensor of second rank, as above.

# Homework of Chap. 6

**Problem 6.11**

A transverse plane wave is incident normally in vacuum on a perfectly absorbing flat screen.

(a)From the law of conservation of linear momentum, show that the pressure (called radiation pressure) exerted on the screen is equal to the field energy per unit volume in the wave.

(b)In the neighborhood of the earth the flux of electromagnetic energy from the sun is approximately 1.4kW/m$^2$. If an interplanetary "sailplane" had a sail of mass1g/m$^2$ of area and negligible other weight, what would be its maximum acceleration in meters per second squared due to the solar radiation pressure? How does this compare with the acceleration due to the solar "wind" (corpuscular radiation)?

**Problem 6.12**

Consider the definition of the admittance $Y = G - iB$ of a two terminal linear passive network in terms of field quantities by means of the complex Poynting theorem of Section 6.9.

(a) By considering the complex conjugate of (6.134) obtain general expressions for the conductance $G$ and susceptance $B$ for the general case including radiation loss.

(b) Show that at low frequencies the expressions equivalent to (6.139) and (6.140) are

$$G \simeq \frac{1}{|V_i|^2} \int_V \sigma |\mathbf{E}|^2 d^3x$$

$$B \simeq -\frac{4\omega}{|V_i|^2} \int_V (w_m - w_e) d^3x$$

# Homework of Chap. 6

**Problem 6.15**

If a conductor or semiconductor has current flowing in it because of an applied electric field, and a transverse magnetic field is applied, there develops a component of electric field in the direction orthogonal to both the applied electric field (direction of current flow) and the magnetic field, resulting in a voltage difference between the sides of the conductor. This phenomenon is known as the *Hall effect*.

(a) Use the known properties of electromagnetic fields under rotations and spatial reflections and the assumption of Taylor series expansions around zero magnetic field strength to show that for an isotropic medium the generalization of Ohm?s law, correct to second order in the magnetic field, must have the form

$$\mathbf{E} = \rho_0 \mathbf{J} + R(\mathbf{H}\times\mathbf{J}) + \beta_1 H^2 \mathbf{J} + \beta_2 (\mathbf{H}\cdot\mathbf{J})\mathbf{H}$$

where $\rho_0$ is the resistivity in the absence of the magnetic field and $R$ is called the Hall coefficient.

(b) What about the requirements of time reversal invariance?

**Problem 6.19**

(a) Apply space inversion to the monopole vector potential of Problem 6.18 and show that the vector potential becomes

$$A_\phi' = -g\frac{(1+\cos\theta)}{4\pi r\sin\theta} = -\frac{g}{4\pi r}\cot\left(\frac{\theta}{2}\right)$$

with the other components vanishing, Show explicitly that its curl gives the magnetic field of a magnetic monopole, except perhaps at $\theta = 0$. [Remember the space-inversion properties of the magnetic charge!]

(b) Show that the difference, $\delta\mathbf{A} = \mathbf{A}'$ ?$\mathbf{A}$, can be expressed as the gradient of a scalar function, indicating that the original and space-inverted vector potentials differ by a gauge transformation.

(c) Interpret the gauge function in terms of Fig. 6.9.[ $H$ int: Choose the contour $C$ to be a rectangle lying in a plane containing the $z$ axis, with three sides at infinity.]

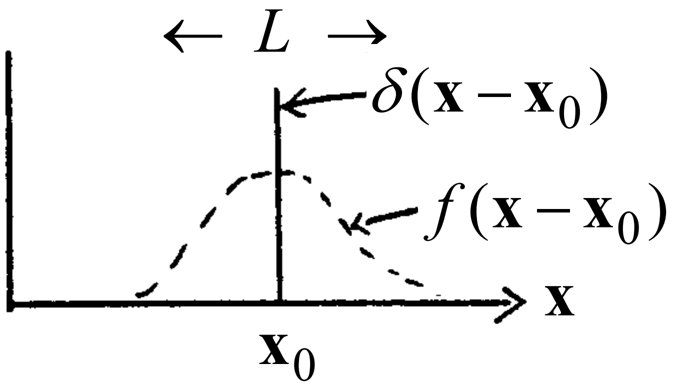
Optional

# 6.6 Derivation of the Equations of Macroscopic Electromagnetism

We limit the scope of our consideration of Sec. 6.6 to a general discussion of the averaging method and the derivation of (6.65).

Microscopically, the matter is composed of electrons and nuclei, in which the spatial variations of charge/current distribution functions and electromagnetic field functions occur over the atomic distances (of the order of $10^{-10}$ m). These functions can be regarded as sums of delta functions. However, macroscopic instruments only measure the averaged quantity. Hence there is a need to develop an averaging method to reduce microscopically fluctuating functions to macroscopically smooth functions, and thereby obtain a set of macroscopic Maxwell equations.

If we replace each delta function, e.g. $\delta(\mathbf{x}-\mathbf{x}_0)$, in the microscopic distribution function (of charges, etc.) with a smooth function $f(\mathbf{x}-\mathbf{x}_0)$ (see figure) subject to the condition




$$\int f(\mathbf{x}-\mathbf{x}_0)d^3x=1$$

and if the width $L$ of $f(\mathbf{x}-\mathbf{x}_0)$ is much greater than the atomic distances (e.g. $L\approx 10^{-8}$ m), then the sum of many such functions (each representing a delta function in the microscopic distribution function) will become a smooth function representing the spatially averaged microscopic distribution function. This is the method used in Sec. 6.6 for the derivation of macroscopic equations.

We may look at the above averaging procedure as follows. A delta function $\delta(\mathbf{x}-\mathbf{x}_0)$ generates a smooth function $f(\mathbf{x}-\mathbf{x}_0)$. Thus, for a distribution function $F(\mathbf{x})$ composed of a large number of point sources (delta functions), the response [denoted by $\langle F(\mathbf{x})\rangle$] will be the superposition of the responses from all points:

$$\langle F(\mathbf{x})\rangle = \int f(\mathbf{x}-\mathbf{x}_0)F(\mathbf{x}_0)d^3x_0 \text{..... spatial average of } F(\mathbf{x})$$

In the integrand, replacing $\mathbf{x}_0$ with $\mathbf{x}-\mathbf{x}'$, we obtain (6.65):

$$\langle F(\mathbf{x})\rangle = \int f(\mathbf{x}')F(\mathbf{x}-\mathbf{x}')d^3x', \tag{6.65}$$

where $f(\mathbf{x})$ is now a smooth function centered at $\mathbf{x}=0$.

As an example, we let $F(\mathbf{x})=\delta(\mathbf{x}-\mathbf{x}_0)$ and sub. it into (6.65)

$$\langle \delta(\mathbf{x}-\mathbf{x}_0)\rangle = \int f(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}_0-\mathbf{x}')d^3x' = f(\mathbf{x}-\mathbf{x}_0)$$

Thus, we have recovered our assumption that the delta function $\delta(\mathbf{x}-\mathbf{x}_0)$ generates a smooth function $f(\mathbf{x}-\mathbf{x}_0)$ centered at $\mathbf{x}_0$.

**Coulomb Gauge :** (also called radiation gauge, transverse gauge, or solenoid gauge)

In the Coulomb gauge, we have $\nabla \cdot \mathbf{A} = 0$ (6.21)

then, 
$$\begin{cases} (6.10) \Rightarrow \nabla^2 \Phi = -\dfrac{\rho}{\varepsilon_0} & (6.22) \\ (6.11) \Rightarrow \nabla^2 \mathbf{A} - \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}\mathbf{A} = -\mu_0 \mathbf{J} + \dfrac{1}{c^2}\nabla \dfrac{\partial \Phi}{\partial t} & (6.24) \end{cases}$$

To uncouple $\mathbf{A}$ and $\Phi$, we write $\mathbf{J} = \mathbf{J}_l + \mathbf{J}_t$ and demand

$$\begin{cases} \nabla \times \mathbf{J}_l = 0 & [\mathbf{J}_l \text{ is called longitudinal or irrotational current}] \\ \nabla \cdot \mathbf{J}_t = 0 & [\mathbf{J}_t \text{ is called transverse or solenoidal current}] \end{cases}$$

We may construct $\mathbf{J}_l$ and $\mathbf{J}_t$ from $\mathbf{J}$ as follows:

$$\begin{cases} \mathbf{J}_l = -\dfrac{1}{4\pi}\nabla \displaystyle\int \dfrac{\nabla' \cdot \mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' & (6.27) \\ \mathbf{J}_t = \dfrac{1}{4\pi}\nabla \times \nabla \times \displaystyle\int \dfrac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' & (6.28) \end{cases}$$

See proof at the end of this section.

Optional 

Rewrite (6.22): $\nabla^2\Phi = -\dfrac{\rho}{\varepsilon_0}$. The solution is

$$\Phi(\mathbf{x},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}' \quad \begin{bmatrix}\text{called the instantaneous} \\ \text{Coulomb potential}\end{bmatrix} \qquad (6.23)$$

In $\mathbf{J}_l = -\frac{1}{4\pi}\nabla\int\frac{\nabla'\cdot\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$, replacing $\nabla\cdot\mathbf{J}$ with $-\dfrac{\partial\rho}{\partial t}$ and use (6.23) and $c^2 = \dfrac{1}{\varepsilon_0\mu_0}$, we obtain

$$\frac{1}{c^2}\nabla\frac{\partial\Phi}{\partial t} = \mu_0\mathbf{J}_l \qquad (6.28)$$

Sub. $\mathbf{J}_l$ from (6.28) into

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} = -\mu_0(\mathbf{J}_l + \mathbf{J}_t) + \frac{1}{c^2}\nabla\frac{\partial\Phi}{\partial t} \qquad (6.24)$$

The last term on the RHS of (6.24) is then cancelled by $\mathbf{J}_l$ to result in an equation for $\mathbf{A}$ uncoupled from $\Phi$:

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} = -\mu_0\mathbf{J}_t \qquad (6.30)$$

*Discussion*:

(i) $\nabla \cdot \mathbf{J}_t = 0 \;\Rightarrow \mathbf{J}_t$ does not lead to time variation of charge density $\rho$ [see (1)].

(ii) $\Phi \propto \frac{1}{r} \Rightarrow \nabla\Phi \propto \frac{1}{r^2} \Rightarrow \begin{cases} \text{1. } \Phi \text{ contributes only to the near fields.} \\ \text{2. Radiation fields are given by } \mathbf{A} \text{ alone.} \\ \text{3. Coulomb gauge allows separation of} \\ \quad \text{"near" and "radiation" fields.} \end{cases}$

(iii) The Coulomb gauge is often used when there is no source.

Then, $\Phi = 0$ and $\mathbf{A}$ satisfies the homogeneous equation

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} = 0.$$

with the fields given by

$$\mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}, \quad \mathbf{B} = \nabla \times \mathbf{A} \qquad (6.31)$$

Optional



*Problem*: Prove $\mathbf{J}_l$ [in (6.27)] $+\ \mathbf{J}_t$[in (6.28)] $= \mathbf{J}$

*Proof*: $$\mathbf{J}_t = \frac{1}{4\pi}\nabla\times\nabla\times\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \qquad (6.28)$$

$$= \frac{1}{4\pi}\Big[\nabla\underbrace{\left(\nabla\cdot\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'\right)}_{\text{(A)}} - \underbrace{\nabla^2\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'}_{\text{(B)}}\Big]$$

$$\text{(A)} = \int\nabla\cdot\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' = \int\mathbf{J}(\mathbf{x}')\cdot\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|}d^3x' = -\int\mathbf{J}(\mathbf{x}')\cdot\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$

$$= \int\frac{\nabla'\cdot\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' - \underbrace{\int\nabla'\cdot\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'}_{0\,\text{(by the divergence thm.)}}$$

$$\text{(B)} = \int\mathbf{J}(\mathbf{x}')\nabla^2\frac{1}{|\mathbf{x}-\mathbf{x}'|}d^3x' = -4\pi\int\mathbf{J}(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')d^3x' = -4\pi\mathbf{J}(\mathbf{x})$$

$$\Rightarrow \mathbf{J}_t = \frac{1}{4\pi}\Big[\underbrace{\nabla\int\frac{\nabla'\cdot\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'}_{-4\pi\mathbf{J}_l\ \text{by (6.27)}} + 4\pi\mathbf{J}(\mathbf{x})\Big] = -\mathbf{J}_l + \mathbf{J} \qquad \text{QED}$$